

2.4 Numerical Approximation: Euler's Method

In this section, we will talk about a numerical procedure for solving ordinary differential equations with a given initial value, which is called Euler method.

Let's look at an example.

Example 1 $\frac{dy}{dx} = y + 1, \quad y(0) = 1 \quad (1)$

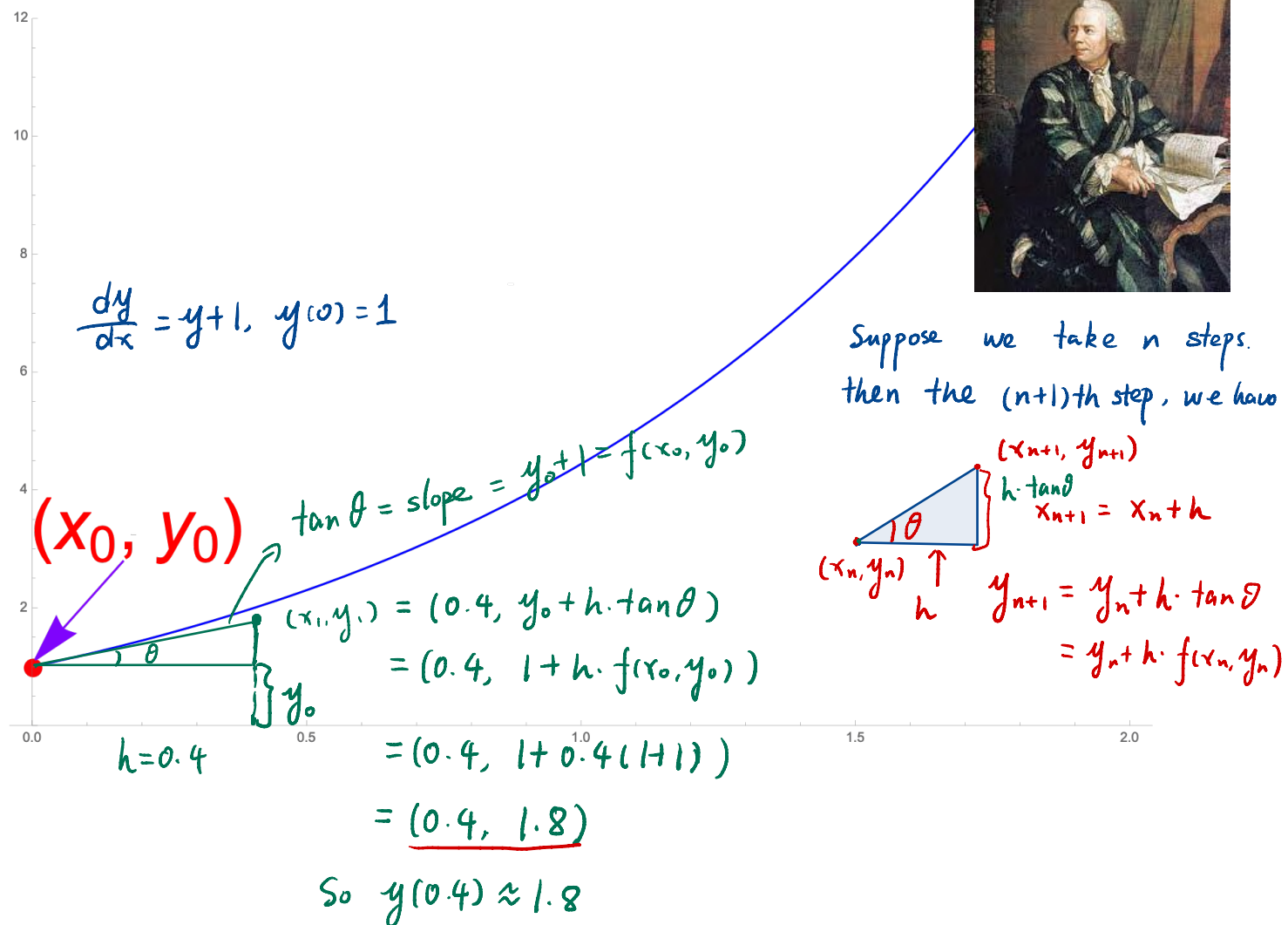
This is a separable differential equation. In practice, we will be given more complicated functions $f(x, y)$ in a more general form:

$$\frac{dy}{dx} = f(x, y), \quad f(x_0) = y_0.$$

Using the method of solving separable equations (Section 1.4), we know the solution of Eq. (1) is $y(x) = 2e^x - 1$. I'll leave this as an exercise for you to check.

Now, let's "pretend" that we don't know how the solution curve look like.

How do we approximate the solution curve with the information given in Eq. (1)? 🤔

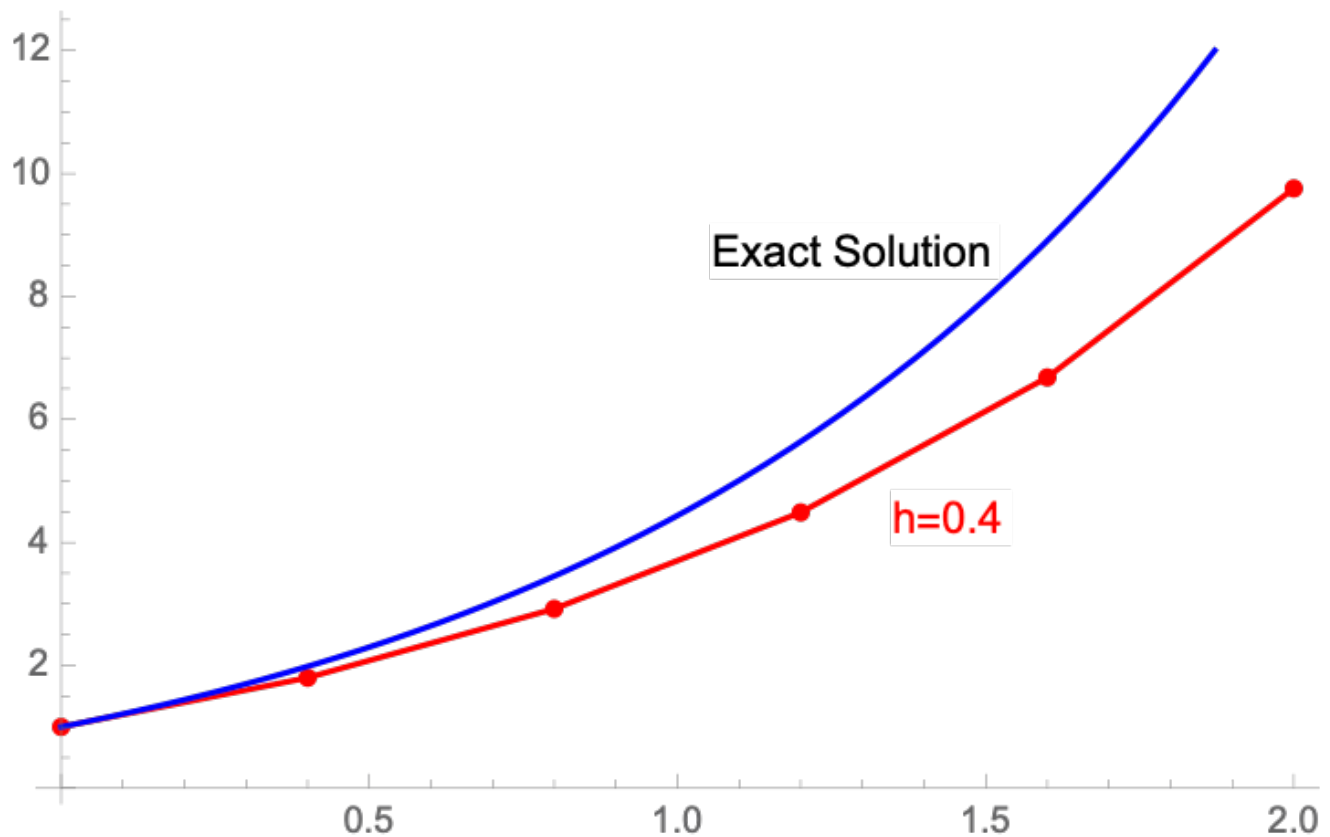


Let $h = 0.4$ (step size). We compute the following table:

	$x_n = x_{n-1} + h$	$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$	(x_n, y_n)
$n = 0$	$x_0 = 0$	$y_0 = 1$	$(0, 1)$
$n = 1$	$x_1 = x_0 + h = 0.4$	$y_1 = y_0 + h \cdot (y_0 + 1) = 1 + 0.4 \cdot 2 = 1.8$	$(0.4, 1.8)$
$n = 2$	$x_2 = x_1 + h = 0.8$	$y_2 = y_1 + h \cdot (y_1 + 1) = 1.8 + 0.4 \cdot 2.8 = 2.92$	$(0.8, 2.92)$
$n = 3$	$x_3 = x_2 + h = 1.2$	$y_3 = y_2 + h \cdot (y_2 + 1) = 4.488$	$(1.2, 4.488)$
$n = 4$	$x_4 = x_3 + h = 1.6$	$y_4 = y_3 + h \cdot (y_3 + 1) = 6.6832$	$(1.6, 6.6832)$
$n = 5$	$x_5 = x_4 + h = 2$	$y_5 = y_4 + h \cdot (y_4 + 1) = 9.75648$	$(2, 9.75648)$

Table 1

Then we plot those points on the xy -plane and connect them as those red straight lines.



How do we increase the accuracy of this approximation?

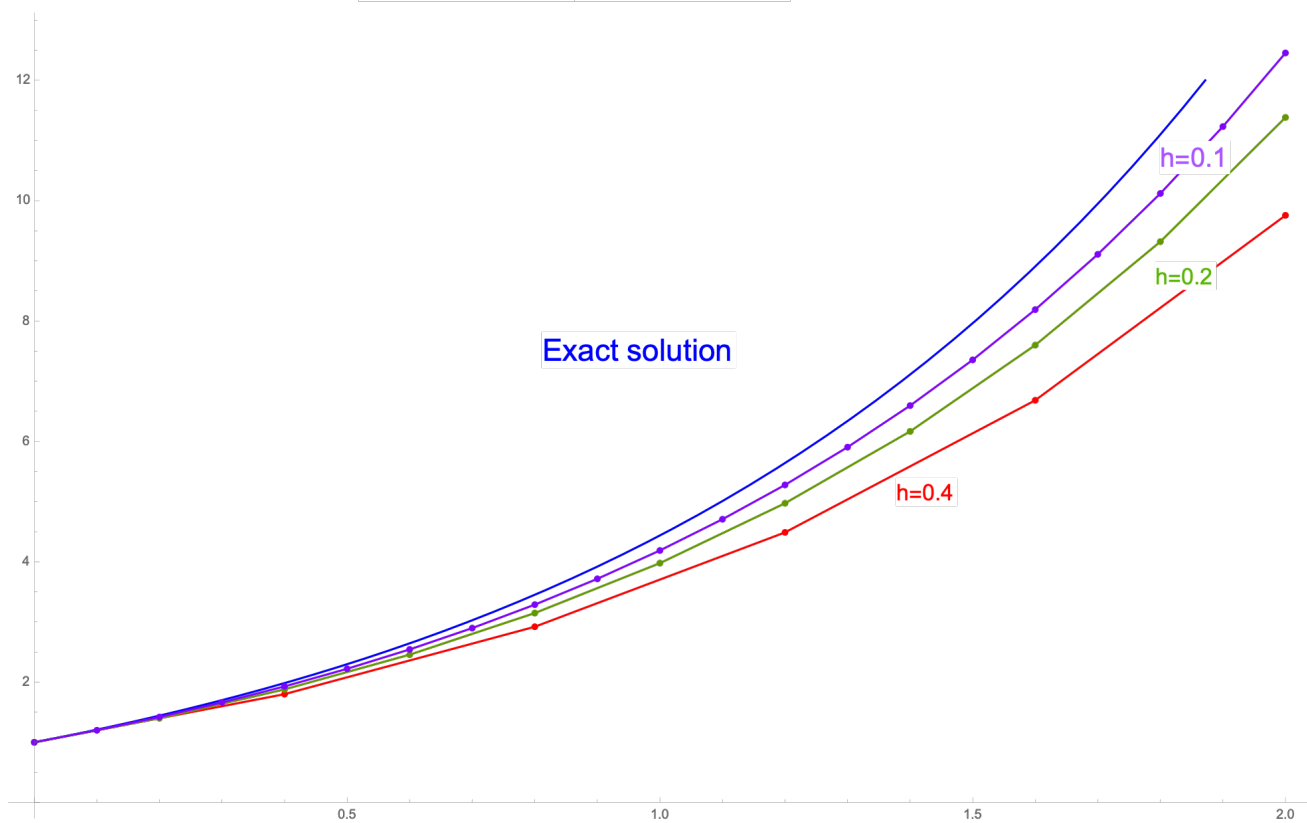
Let's try smaller step size $h = 0.2$, and compute the points similar to **Table 1**. We have the following points:

$h=0.2$

(0,1)
(0.2, 1.4)
(0.4, 1.88)
(0.6, 2.456)
(0.8, 3.1472)
(1, 3.97664)
(1.2, 4.971968)
(1.4, 6.1663616)
(1.6, 7.59963392)
(1.8, 9.319560704)
(2, 11.3834728448)

$h=0.1$

0	1
0.1	1.2
0.2	1.42
0.3	1.662
0.4	1.9282
0.5	2.22102
0.6	2.543122
0.7	2.8974342
0.8	3.28717762
0.9	3.715895382
1	4.1874849202
1.1	4.70623341222
1.2	5.276856753442
1.3	5.9045424287862
1.4	6.59499667166482
1.5	7.3544963388313
1.6	8.18994597271443
1.7	9.10894056998588
1.8	10.1198346269845
1.9	11.2318180896829
2	12.4549998986512



Algorithm The Euler Method

Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

Euler's method with step size h consists of applying the iterative formula $x_{n+1} = x_n + h$

$$y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad (n \geq 0)$$

to calculate successive approximations y_1, y_2, y_3, \dots to the [true] values $y(x_1), y(x_2), y(x_3), \dots$ of the [exact] solution $y = y(x)$ at the points x_1, x_2, x_3, \dots , respectively.

Example 2 The exact solution $y(x)$ is given for the initial value problem. Apply Euler's method twice to approximate to this solution on the interval $[0, \frac{1}{2}]$, first with step size $h = 0.25$, then with step size $h = 0.1$. Compare the three-decimal-place values of the two approximations at $x = \frac{1}{2}$ with the value $y(\frac{1}{2})$ of the actual solution.

$$y' = -3x^2y, \quad y(0) = 3; \quad y(x) = 3e^{-x^3}$$

$$h = 0.25 = \frac{1}{4}$$

$n=0$	$x_0 = 0$	$y_0 = 3$
$n=1$	$x_1 = 0.25$	$y_1 = y_0 + h \cdot f(x_0, y_0) = 3 + 0.25 \cdot (-3 \cdot 0^2 \cdot 3) = 3$
$n=2$	$x_2 = 0.5$	$y_2 = y_1 + h \cdot f(x_1, y_1) = 3 + 0.25 \cdot (-3 \cdot 0.25^2 \cdot 3) = 2.859375$

$$y_{app1}(\frac{1}{2}) = 2.859$$

$$h = 0.1$$

$n=0$	$x_0 = 0$	$y_0 = 3$
$n=1$	$x_1 = 0.1$	$y_1 = y_0 + h \cdot f(x_0, y_0) = 3 + 0.1 \cdot 0 = 3$
$n=2$	$x_2 = 0.2$	$y_2 = y_1 + h \cdot f(x_1, y_1) = 3 + 0.1 \cdot (-3 \cdot 0.1^2 \cdot 3) = 2.991$
$n=3$	$x_3 = 0.3$	$y_3 = y_2 + h \cdot f(x_2, y_2) = 2.955108$
$n=4$	$x_4 = 0.4$	$y_4 = y_3 + h \cdot f(x_3, y_3) = 2.875320$
$n=5$	$x_5 = 0.5$	$y_5 = y_4 + h \cdot f(x_4, y_4) = 2.73730$

$$y_{app2}(\frac{1}{2}) = 2.73730$$

$$y_{actual} = 3 \cdot e^{-(\frac{1}{2})^3}$$

$$\approx 2.647491$$

Here we show how to use computer programs for Euler's method. You can find the .m file in Brightspace with the filled-in notes.

In **Matlab**, we write two .m files like the following. You can also find this in your text book Page 115.

1. First we define the function **euler1.m** like the following. We need to save the file name as

```
function [X, Y] = euler1(x,y,x1,n)
h = (x1 - x)/n; %step size
X = x;          % initial x
Y = y;          %initial y
for i = 1:n %begin loop
    y = y + h*f(x,y); %Euler iteration
    x = x + h; %new x
    X = [X;x]; %update x-column
    Y = [Y;y]; %update y-column
end %end loop
```

2. Then we define our function yp.m below. We save this as f.m

```
function yp = f(x,y)
yp = -3*x^2*y; %yp=y'
```

In the command line, we type

```
>> [X,Y]=euler1(0,3,0.5,2)

X =
    0
    0.2500
    0.5000
Y =
    3.0000
    3.0000
    2.8594

>> [X,Y]=euler1(0,3,0.5,5)
X =
    0
    0.1250
    0.2500
    0.3750
    0.5000
Y =
```

```
3.0000
3.0000
2.9824
2.9125
2.7589
```

Example 3 Use Euler's method with a computer system to find the desired solution values in the following problem. Start with step size $h = 0.1$, and then use successively smaller step sizes until successive approximate solution values at $x = 2$ agree rounded off to two decimal places.

$$y' = x + \frac{1}{2}y^2, y(-2) = 0; \quad y(2) = ?$$

euler1.m

We will solve this problem using Matlab. Please write the *euler.m* file in the previous example and save it first.

Now we write the following function f.m file

```
function yp = f(x,y)
yp = x+(1/2)*y^2; %yp=y'
```

Then we run the following command in the command window. Notice that $h = 0.1$ means we take $n = (2 - (-2))/0.1 = 40$ steps

euler1

```
>> [X,Y]=euler(-2,0,2,40)
```

The output is very long so I just record the last value of the vector Y

```
Y=
0
%a very long list

1.2900
```

We run the program again with $n = 160$

```
Y =  
  
      0  
%long list  
      1.4150
```

We run the program again with $n = 640$

```
Y =  
  
      0  
%long list  
      1.4509
```

We run the program again with $n = 1280$

```
Y =  
  
      0  
%long list  
      1.4571
```

We keep running with smaller h (by increasing n) and we find

$n = 2560, h = 0.0016, y(2) = 1.4602$ and $n = 5120, h = 0.0008, y(2) = 1.4617$. They agree when we rounded off to two decimal places. So $y(2) = 1.46$ rounded off accurate to 2 decimal places.